Minimal quantities and measurability in quantum theory and thermodynamics. Some implications

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The main target of these works is to construct a correct quantum theory and gravity in terms of the variations (increments) dependent on the existent energies

By the author's opinion, these problems are solvable but beyond the paradigm of continuous space-time.

1. **Minimal Length and the Existence of Some Infinitesimal Quantities in Quantum Theory and Gravity,**
   Advances in High Energy Physics, Volume 2014 (2014), Article ID 195157, 8 pages
2. **Minimal Length, Measurability and Gravity,**
   ***Entropy* 2016, *18*(3), 80; doi:10.3390/e18030080**

3. **3 papers in Adv.Stud. Theor. Phys; 2 papers NPCS, 1 papers *J. of Adv. Phys;***

3 papers in International Collections

The principal idea of this papers is as follows:

(1.1) Within a discrete model for continuous space-time, at low energies (which are far from the Planck energies) the results, to a high accuracy, are identical to those obtained by a continuous model for space-time (and in this case may be called the quasi-continuous model). But at high (Planck's) energies the indicated model is fundamentally discrete, leading to principally new results.

(1.2) All variations in any physical system considered in such a discrete model should be dependent on the existent energies.

Minimal Length, Minimal Inverse Temperature, and Measurability

Generalized Uncertainty Principles in Quantum Theory and Thermodynamics

Heisenberg Uncertainty Principle (relation) for momentum - coordinate:

$$\boldsymbol{\Delta x} \geq \frac{\hbar}{\boldsymbol{\Delta p}}. \quad (2)$$

It was shown that at the Planck scale a high-energy term must appear:

$$\boldsymbol{\Delta x} \geq \frac{\hbar}{\boldsymbol{\Delta p}} + \boldsymbol{\alpha' l_p^2} \frac{\boldsymbol{\Delta p}}{\hbar} \quad (3)$$

where $\boldsymbol{l_p}$ is the Planck length $\boldsymbol{l_p^2} = G\hbar/c^3; 1{,}6\ \ 10^{-35}m$ and α' is a constant. Relation (3) is quadratic in $\boldsymbol{\Delta p}$

$$\boldsymbol{\alpha' l_p^2} \, (\boldsymbol{\Delta p})^2 - \hbar \, \boldsymbol{\Delta x \Delta p} + \hbar^2 \leq \mathbf{0} \qquad (4)$$

and therefore leads to the fundamental length

$$\boldsymbol{\Delta x_{min}} = \mathbf{2}\sqrt{\boldsymbol{\alpha'}}\, \boldsymbol{l_p} \quad (5)$$

(3) is called the Generalized Uncertainty Principle (GUP) in Quantum Theory.

(3) gives

$$\frac{\Delta x}{c} \geq \frac{\hbar}{\Delta p c} + \alpha' l_p^2 \frac{\Delta p}{c\hbar}, \qquad \textbf{(6)}$$

then

$$\Delta t \geq \frac{\hbar}{\Delta E} + \alpha' t_p^2 \frac{\Delta E}{\hbar}. \qquad \textbf{(7)}$$

where t_p is the Planck time $t_p = l_p/c = \sqrt{G\hbar/c^5}; 0{,}54 \ \ 10^{-43} sec.$

$$t_{min} = 2\sqrt{\alpha'} t_p \qquad \textbf{(8)}$$

Thus, the inequalities discussed can be rewritten in a standard form

$$\begin{cases} \Delta x \geq \dfrac{\hbar}{\Delta p} + \alpha' \left(\dfrac{\Delta p}{P_{pl}}\right) \dfrac{\hbar}{P_{pl}} \\ \Delta t \geq \dfrac{\hbar}{\Delta E} + \alpha' \left(\dfrac{\Delta E}{E_p}\right) \dfrac{\hbar}{E_p} \end{cases} \qquad \textbf{(9)}$$

where $P_{pl} = E_p/c = \sqrt{\hbar c^3/G}$.

Now we consider the thermodynamics uncertainty relations between the inverse temperature and interior energy of a macroscopic ensemble

$$\boldsymbol{\Delta \frac{1}{T} \geq \frac{k_B}{\Delta U}}, \quad (k_B \text{ is the Boltzmann constant}) \qquad \textbf{(10)}$$

(Uncertainty Principle in Thermodynamics **UPT**)

It is obvious from the above inequalities that at very high energies the capacity of the heat bath can no longer to be assumed infinite at the Planck scale. Indeed, the total energy of the pair heat bath - ensemble may be arbitrary large but finite merely as the universe is born at a finite energy. Hence the quantity that can be interpreted as the temperature of the ensemble must have the upper limit and so does its main quadratic deviation. In other words the quantity $\Delta(1/T)$ must be bounded from below. But in this case an additional term should be introduced into **(10)**

$$\boldsymbol{\Delta \frac{1}{T} \geq \frac{k_B}{\Delta U} + \eta\, \Delta U}, \ (\boldsymbol{\eta} \text{ is a coefficient}) \qquad \textbf{(11)}$$

(Generalized Uncertainty Principle in Thermodynamics -- **GUPT**)

Dimension and symmetry reasons give

$$\boldsymbol{\eta = \frac{k_B}{E_p^2} \quad or \quad \eta = \alpha' \frac{k_B}{E_p^2}} \qquad \textbf{(12)}$$

As in the previous cases inequality **(11)** leads to the fundamental (inverse) temperature.

$$\boldsymbol{T_{max} = \frac{\hbar}{2\sqrt{\alpha'} t_p k_B} = \frac{E_p}{2\sqrt{\alpha'} k_B} = \frac{T_p}{2\sqrt{\alpha'}} = \frac{\hbar}{t_{min} k_B},} \qquad \textbf{(13)}$$
$$\boldsymbol{\beta_{min} = \frac{1}{k_B T_{max}} = \frac{t_{min}}{\hbar}}$$

Thus, we obtain the system of generalized uncertainty relations in a symmetric form

$$\begin{cases} \boldsymbol{\Delta x} & \geq \boldsymbol{\frac{\hbar}{\Delta p} + \alpha' \left(\frac{\Delta p}{P_{pl}}\right) \frac{\hbar}{P_{pl}} + \ldots} \\ \boldsymbol{\Delta t} & \geq \boldsymbol{\frac{\hbar}{\Delta E} + \alpha' \left(\frac{\Delta E}{E_p}\right) \frac{\hbar}{E_p} + \ldots} \\ \boldsymbol{\Delta \frac{1}{T}} & \geq \boldsymbol{\frac{k_B}{\Delta U} + \alpha' \left(\frac{\Delta U}{E_p}\right) \frac{k_B}{E_p} + \ldots} \end{cases} \qquad \textbf{(14)}$$

or in the equivalent form

$$\begin{cases} \boldsymbol{\Delta x} & \geq & \frac{\hbar}{\boldsymbol{\Delta p}} + \boldsymbol{\alpha' l_p^2} \frac{\boldsymbol{\Delta p}}{\hbar} + \ldots \\ \boldsymbol{\Delta t} & \geq & \frac{\hbar}{\boldsymbol{\Delta E}} + \boldsymbol{\alpha' t_p^2} \frac{\boldsymbol{\Delta E}}{\hbar} + \ldots \\ \boldsymbol{\Delta \frac{1}{T}} & \geq & \frac{\boldsymbol{k_B}}{\boldsymbol{\Delta U}} + \boldsymbol{\alpha'} \frac{\boldsymbol{1}}{\boldsymbol{T_p^2}} \frac{\boldsymbol{\Delta U}}{\boldsymbol{k_B}} + \ldots \end{cases} \quad \textbf{(15)}$$

where the dots mean the existence of higher order corrections.

Here T_p is the Planck temperature: $\boldsymbol{T_p = E_p/k_B}$.

In this case, without the loss of generality and for symmetry, it is assumed that a dimensionless constant in the right-hand side of **GUP** and in the right-hand side of **GUPT** is the same -- α'.

Minimal Length and Measurable Notion in Quantum Theory

Definition I. *Let us call as* ***primarily measurable variation*** *any small variation (increment)* $\tilde{\Delta}x_\mu$ *of any spatial coordinate* x_μ *of the arbitrary point* $x_\mu, \mu = 1,\ldots,3$ *in some space-time system* R *if it may be realized in the form of the uncertainty (standard deviation)* Δx_μ *when this coordinate is measured within the scope of Heisenberg's Uncertainty Principle (HUP):*

$$\widetilde{\boldsymbol{\Delta}}\boldsymbol{x}_{\boldsymbol{\mu}} = \boldsymbol{\Delta x}_{\boldsymbol{\mu}}, \boldsymbol{\Delta x}_{\boldsymbol{\mu}} = \frac{\hbar}{\boldsymbol{\Delta p}_{\boldsymbol{\mu}}}, \boldsymbol{\mu} = \mathbf{1}, \mathbf{2}, \mathbf{3} \qquad (2a)$$

for some $\Delta p_\mu \neq 0$*.*

Similarly, for $\mu = 0$ *for pair ``time-energy''* (t, E)*, let's call any small variation (increment) by* ***primarily measurable variation*** *in the value of time* $\tilde{\Delta}x_0 = \tilde{\Delta}t_0$ *if it may be realized in the form of the uncertainty (standard deviation)* $\Delta x_0 = \Delta t$ *and then*

$$\widetilde{\boldsymbol{\Delta}}\boldsymbol{t} = \boldsymbol{\Delta t}, \boldsymbol{\Delta t} = \frac{\hbar}{\boldsymbol{\Delta E}} \qquad (2b)$$

for some $\Delta E \neq 0$*.*

Here HUP is given for the nonrelativistic case. In the relativistic case HUP has the distinctive features which, however, are of no significance for the general formulation of **Definition I.**

It is clear that at low energies $E \ll E_P$ (momenta $P \ll P_{pl}$) **Definition I.** sets a lower bound for the **primarily measurable variation** $\widetilde{\Delta} x_\mu$ of any space-time coordinate x_μ.
At high energies E (momenta P) this is not the case if E (P) have no upper limit. But, according to the modern knowledge, E (P) are bounded by some maximal quantities E_{max}, (P_{max})

$$\boldsymbol{E \leq E_{max}, P \leq P_{max},} \qquad \textbf{(16)}$$

where in general E_{max}, P_{max} may be on the order of Planck quantities $E_{max} \propto E_P, P_{max} \propto P_{pl}$ and also may be the trans-Planck's quantities.
In any case the quantities P_{max} and E_{max} lead to the introduction of the minimal length l_{min} and of the minimal time t_{min}.
Supposition II. *There is the minimal length l_{min} as a minimal measurement unit for all **primarily measurable variations** having the dimension of length, whereas the minimal time $t_{min} = l_{min}/c$ as a minimal measurement unit for all quantities or **primarily measurable variations (increments)** having the dimension of time, where c is the speed of light.*

For definiteness, we consider that E_{max} and P_{max} are the quantities on the order of the Planck quantities, then l_{min} and t_{min} are also on the order of Planck quantities $l_{min} \propto l_P, t_{min} \propto t_P$.

Definition I. and **Supposition II.** are quite natural in the sense that there are no physical principles with which they are inconsistent.

The combination of ***Definition I.*** *and* ***Supposition II.*** *will be called the* ***Principle of Bounded Primarily Measurable Space-Time Variations (Increments)*** *or for short* ***Principle of Bounded Space-Time Variations (Increments)*** *with abbreviation (PBSTV).*

As the minimal unit of measurement l_{min} is available for all the **primarily measurable variations** ΔL having the dimensions of length, the ``Integrality Condition'' (IC) is the case

$$\boldsymbol{\Delta L = N_{\Delta L} l_{min}}, \quad N_{\Delta L} > 0 \text{ is an integer number} \qquad \textbf{(17a)}$$

In a like manner the same ``Integrality Condition'' (IC) is the case for all the **primarily measurable variations** Δt having the dimensions of time. And similar for any time Δt:

$$\boldsymbol{\Delta t = N_{\Delta t} t_{min}}, \qquad \textbf{(17b)}$$

Definition 1 (**Primary or Elementary Measurability.**)
(1) *In accordance with the PBSTV let us define the quantity having the dimensions of length* $\boldsymbol{l}$ *or time* $\boldsymbol{t}$ *as* **primarily (or elementarily) measurable**, *when it satisfies the relation Eqs.* **(17a)** (**(17b)**).
(2) *Let us define any physical quantity* **primarily (or elementarily) measurable**, *when its value is consistent with points* **(1)** *of this Definition.*

It is convenient to use the deformation parameter α_a. This is a *deformation parameter* on going from the canonical quantum mechanics to the quantum mechanics at Planck's scales (early Universe) that is considered to be the quantum mechanics with the minimal length (QMML):

$$\boldsymbol{\alpha_a = l_{min}^2/a^2}, \boldsymbol{a} \textit{ is the measuring scale} \qquad \textbf{(18)}$$

$$\boldsymbol{\alpha_a = l_{min}^2/a^2 = \frac{l_{min}^2}{N_a^2 l_{min}^2} = \frac{1}{N_a^2}}. \qquad \textbf{(19)}$$

It is evident that α_a is irregularly discrete.
It should be noted that, physical **primarily measurable quantities** won't be enough for the research of physical systems.

Indeed, such a variable as

$$\boldsymbol{\alpha_{N_a l_{min}}(N_a l_{min}) = p(N_a)\frac{l_{min}^2}{\hbar} = l_{min}/N_a,} \qquad \textbf{(20)}$$

where $\boldsymbol{\alpha_{N_a l_{min}}} = \boldsymbol{\alpha_a}$ at $a = N_a l_{min}$, $\boldsymbol{p(N_a)} = \frac{\hbar}{\boldsymbol{N_a l_{min}}}$ is the corresponding **primarily measurable momentum**), is fully expressed in terms *only* **Primarily Measurable Quantities** of **Definition 1** and that's why it may appear at any stage of calculations, but apparently doesn't comply with **Definition 1**. That's why it's necessary to introduce the following definition generalizing **Definition 1**:

Definition 2. Generalized Measurability
We shall call any physical quantity as **generalized-measurable** or for simplicity **measurable** if any of its values may be obtained in terms of **Primarily Measurable Quantities** of **Definition 1**.

In what follows, for simplicity, we will use the term **Measurability** instead of **Generalized Measurability**.
It is evident that any **primarily measurable quantity (PMQ)** is **measurable**.
Generally speaking, the contrary is not correct.

The **generalized-measurable** quantities are appeared from the **Generalized Uncertainty Principle (GUP)**:

$$\boldsymbol{\Delta x_{min} = 2\sqrt{\alpha'}\, l_p = l_{min},} \qquad \textbf{(21)}$$

For convenience, we denote the minimal length $l_{min} \neq 0$ by ℓ and $t_{min} \neq 0$ by $\tau = \ell/c$.
Solving nequality $\alpha' l_p^2\,(\Delta p)^2 - \hbar\,\Delta x \Delta p + \hbar^2 \leq 0$, in the case of equality we obtain the apparent formula

$$\boldsymbol{\Delta p_{\pm} = \frac{\left(\Delta x \pm \sqrt{(\Delta x)^2 - 4\alpha' l_p^2}\right)\hbar}{2\alpha' l_p^2}.} \qquad \textbf{(22)}$$

Next, into this formula we substitute the right-hand part of formula **(17a)** for $L = x$. Then:

$$\boldsymbol{\Delta p_{\pm} = \frac{\left(N_{\Delta x} \pm \sqrt{(N_{\Delta x})^2 - 1}\right)\hbar \ell}{\frac{1}{2}\ell^2} = \frac{2\left(N_{\Delta x} \pm \sqrt{(N_{\Delta x})^2 - 1}\right)\hbar}{\ell}.} \qquad \textbf{(23)}$$

But it is evident that at low energies $E \ll E_p; N_{\Delta x} \gg 1$ the plus sign in the nominator **(23)** leads to the contradiction as it results in very high (much greater than the Planck's) values of Δp. Because of this, it is necessary to select the minus sign in the numerator **(23)**. Then, multiplying the left and right sides of **(23)** by the same number $N_{\Delta x} + \sqrt{N_{\Delta x}^2 - 1}$, we get

$$\boldsymbol{\Delta p = \frac{2\hbar}{(N_{\Delta x} + \sqrt{N_{\Delta x}^2 - 1})\ell}}. \qquad \textbf{(23a)}$$

Δp from formula **(23a)** is the **generalized-measurable** quantity in the sense of **Definition 2.** However, it is clear that at low energies $E \ll E_p$, i.e. for $N_{\Delta x} \gg 1$, we have $\sqrt{N_{\Delta x}^2 - 1} \approx N_{\Delta x}$. Moreover, we have

$$\boldsymbol{\lim_{N_{\Delta x} \to \infty} \sqrt{N_{\Delta x}^2 - 1} = N_{\Delta x}}. \qquad \textbf{(24)}$$

Therefore, in this case **(23a)** may be written as follows:

$$\boldsymbol{\Delta p} = \boldsymbol{\Delta p}(\boldsymbol{N_{\Delta x}}, \boldsymbol{HUP}) = \frac{\hbar}{\mathbf{1/2}(\boldsymbol{N_{\Delta x}}+\sqrt{\boldsymbol{N_{\Delta x}^2}-\mathbf{1}})\boldsymbol{\ell}} \approx \frac{\hbar}{\boldsymbol{N_{\Delta x}\ell}} = \frac{\hbar}{\boldsymbol{\Delta x}}; \boldsymbol{N_{\Delta x}} \gg \mathbf{1}$$

(23.HUP)

in complete conformity with HUP. Besides, $\Delta p = \Delta p(N_{\Delta x}, HUP)$, to a high accuracy, is a **primarily measurable** quantity in the sense of **Definition 1**.

And vice versa it is obvious that at high energies $E \approx E_p$, i.e. for $N_{\Delta x} \approx 1$, we can write

$$\boldsymbol{\Delta p} = \boldsymbol{\Delta p}(\boldsymbol{N_{\Delta x}}, \boldsymbol{GUP}) = \frac{\hbar}{\mathbf{1/2}(\boldsymbol{N_{\Delta x}}+\sqrt{\boldsymbol{N_{\Delta x}^2}-\mathbf{1}})\boldsymbol{\ell}}; \boldsymbol{N_{\Delta x}} \approx \mathbf{1}. \qquad \textbf{(23GUP)}$$

At the same time, $\Delta p = \Delta p(N_{\Delta x}, GUP)$ is a **Generalized Measurable** quantity in the sense of **Definition 2**.

Thus, we have

$$\boldsymbol{GUP} \rightarrow \boldsymbol{HUP} \qquad \textbf{(25a)}$$

for

$$(\boldsymbol{N_{\Delta x}} \approx \mathbf{1}) \rightarrow (\boldsymbol{N_{\Delta x}} \gg \mathbf{1}).$$

Also, we have

$$\boldsymbol{\Delta p}(\boldsymbol{N_{\Delta x}}, \boldsymbol{GUP}) \rightarrow \boldsymbol{\Delta p}(\boldsymbol{N_{\Delta x}}, \boldsymbol{HUP}), \qquad \textbf{(25b)}$$

Comment 2.*

From the above formulae it follows that, within GUP, the **primarily measurable** *variations (quantities) are derived to a high accuracy from the* **generalized-measurable** *variations (quantities) only in the low-energy limit* $E \ll E_P$

Next, within the scope of GUP, we can correct a value of the parameter α_a from substituting a for Δx in the expression $N_{\Delta x} + \sqrt{N_{\Delta x}^2 - 1}$.

Then at low energies $E \ll E_p$ we have the **primarily measurable** quantity $\alpha_a(HUP)$

$$\boldsymbol{\alpha_a = \alpha_a(HUP) = \frac{1}{[1/2(N_a + \sqrt{N_a^2 - 1})]^2} \approx \frac{1}{N_a^2}; N_a \gg 1,} \qquad \textbf{(26a)}$$

Accordingly, at high energies we have $E \approx E_p$

$$\boldsymbol{\alpha_a = \alpha_a(GUP) = \frac{1}{[1/2(N_a + \sqrt{N_a^2 - 1})]^2}; N_a \approx 1.} \qquad \textbf{(26b)}$$

When going from high energies $E \approx E_p$ to low energies $E \ll E_p$, we can write

$$\boldsymbol{\alpha_a(GUP) \overset{(N_a \approx 1) \to (N_a \gg 1)}{\to} \alpha_a(HUP)} \qquad \textbf{(26c)}$$

in complete conformity to *Comment 2*.*

Minimal Inverse Temperature and Measurability. Duality

Now, let us return to the thermodynamic relation in the case of equality:

$$\boldsymbol{\Delta \frac{1}{T} = \frac{k_B}{\Delta U} + \eta\, \Delta U}, \quad \textbf{(27a)}$$

that is equivalent to the quadratic equation

$$\boldsymbol{\eta\, (\Delta U)^2 - \Delta \frac{1}{T} \Delta U + k_B = 0}. \qquad \textbf{(27b)}$$

The discriminant of this equation is equal to

$$\boldsymbol{D = (\Delta \frac{1}{T})^2 - 4\eta k_B = (\Delta \frac{1}{T})^2 - 4\alpha' \frac{k_B^2}{E_p^2} \geq 0}, \qquad \textbf{(27c)}$$

leading directly to $(\Delta \frac{1}{T})_{min}$

$$\boldsymbol{(\Delta \frac{1}{T})_{min} = 2\sqrt{\alpha'}\, \frac{k_B}{E_p}} \qquad \textbf{(27d)}$$

or, due to the fact that k_B is constant, we have

$$\boldsymbol{(\Delta \frac{1}{k_B T})_{min} = \frac{2\sqrt{\alpha'}}{E_p}}. \textbf{(27e)}$$

It is clear that $(\Delta \frac{1}{T})_{min}$ corresponds to T_{max}

$$\boldsymbol{T_{max} \approx T_p \gg 0}. \quad \textbf{(27f)}$$

In this case $\Delta \frac{1}{T} \approx \frac{1}{T}$ and, of course, we can assume that

$$\boldsymbol{(\frac{1}{T})_{min} = \tilde{\tau} = \frac{1}{T_{max}}}. \qquad \textbf{(27g)}$$

Trying to find from formula **(27g)** a minimal unit of measurability for the inverse temperature and introducing the ``Integrality Condition'' (IC)

$$\frac{1}{T} = N_{1/T}\tilde{\tau}, N_{1/T} > 0 \text{ is an integer number} \qquad \textbf{(27h)}$$

analog of the **primary measurability** notion into thermodynamics.

Definition 3 (**Primary Thermodynamic Measurability**)
(1) *Let us define a quantity having the dimensions of inverse temperature as* **primarily measurable** *when it satisfies the relation* **(27h)**.
(2) Let us define any physical quantity in thermodynamics as **primarily measurable** when its value is consistent with point (1) of this Definition.

Definition 3 in thermodynamics is analogous to the **Primary Measurability** in a quantum theory (**Definition 1**).
Now we consider the quadratic equation **(27b)** in terms of **measurable quantities** in the sense of **Definition 3**. In accordance with this definition we can write

$$\boldsymbol{\Delta\frac{1}{T} = N_{\Delta(1/T)}\tilde{\tau},\ \ N_{\Delta(1/T)} > 0} \text{ is an integer number} \qquad \textbf{(28a)}$$

This quadratic equation takes the following form:

$$\boldsymbol{\eta\,(\Delta U)^2 - N_{\Delta(1/T)}\tilde{\tau}\Delta U + k_B = 0.} \qquad \textbf{(28b)}$$

we can find the '' **measurable**'' roots of this equation:

$$(\boldsymbol{\Delta U})_{meas,\pm} = \frac{[N_{\Delta(1/T)} \pm \sqrt{N^2_{\Delta(1/T)} - 1}]\tilde{\tau}}{2\eta} =$$

$$= \frac{2k_B[N_{\Delta(1/T)} \pm \sqrt{N^2_{\Delta(1/T)} - 1}]\tilde{\tau}}{\tilde{\tau}^2} = \qquad \textbf{(28c)}$$

$$\frac{2k_B[N_{\Delta(1/T)} \pm \sqrt{N^2_{\Delta(1/T)} - 1}]}{\tilde{\tau}}.$$

The last line in is associated with the obvious relation $2\eta = \frac{\tilde{\tau}^2}{2k_B}$.

In this way we derive a complete analog of the corresponding relation from a quantum theory by replacement

$$\boldsymbol{\Delta p_{\pm} \Rightarrow \Delta U_{meas,\pm}; N_{\Delta x} \Rightarrow N_{\Delta(1/T)}; \hbar \Rightarrow k_B}. \qquad \textbf{(28d)}$$

As, for **low temperatures and energies**, $T \ll T_{max} \propto T_p$, we have $1/T \gg 1/T_p$ and hence $\Delta(1/T) \gg 1/T_p$ and $N_{\Delta(1/T)} \gg 1$.
Next, in analogy with Subsection 2.2, we can have only the minus-sign root, otherwise, at sufficiently high $N_{\Delta(1/T)} \gg 1$ for $(\Delta U)_{meas,+}$ we can get $(\Delta U)_{meas,+} \gg E_p$. But this is impossible for low temperatures (energies).

On the contrary, the minus sign in **(28c)** is consistent with high and low energies. So, taking the root value in **(28c)** corresponding to this sign and multiplying the nominator and denominator in **(28c)** by $N_{\Delta(1/T)} + \sqrt{N^2_{\Delta(1/T)} - 1}$, we obtain

$$(\boldsymbol{\Delta U})_{\boldsymbol{meas}} = \frac{\boldsymbol{k_B}}{\frac{1}{2}(\boldsymbol{N_{\Delta(1/T)}} + \sqrt{\boldsymbol{N^2_{\Delta(1/T)}} - \mathbf{1}})\tilde{\boldsymbol{\tau}}} \qquad \textbf{(28e)}$$

to have a complete analog of the corresponding relation from quantum theory by substitution.
Then it is clear that, in analogy with QT, for low energies and temperatures $N_{\Delta(1/T)} \gg 1$ may be rewritten as

$$(\boldsymbol{\Delta U})_{\boldsymbol{meas}} = (\boldsymbol{\Delta U})_{\boldsymbol{meas}}(\boldsymbol{T} \ll \boldsymbol{T_{max}}) = \frac{\boldsymbol{k_B}}{\frac{1}{2}(\boldsymbol{N_{\Delta(1/T)}} + \sqrt{\boldsymbol{N^2_{\Delta(1/T)}} - \mathbf{1}})\tilde{\boldsymbol{\tau}}} \approx$$

$$\approx \frac{\boldsymbol{k_B}}{\boldsymbol{N_{\Delta(1/T)}}\tilde{\boldsymbol{\tau}}}, \boldsymbol{N_{\Delta(1/T)}} \gg \mathbf{1},$$

(28f)

i.e. the Uncertainty Principle in Thermodynamics (UPT) is involved. In this case, due to the last formula, ΔU_{meas} represents a **primarily measurable** thermodynamic quantity in the sense of **Definition 3** to a high accuracy.

Of course, at high energies the last term in the formula **(28f)** is lacking and, for $T \approx T_{max}; N_{\Delta(1/T)} \approx 1$, we have:

$$(\boldsymbol{\Delta U})_{meas} = (\boldsymbol{\Delta U})_{meas}(\boldsymbol{T} \approx \boldsymbol{T}_{max}) = \frac{\boldsymbol{k_B}}{\mathbf{1/2}(\boldsymbol{N}_{\Delta(1/T)}+\sqrt{\boldsymbol{N}^2_{\Delta(1/T)}-\mathbf{1}})\tilde{\boldsymbol{\tau}}},$$

$$\boldsymbol{N}_{\Delta(1/T)} \approx \mathbf{1}.$$

(28g)

From **(28g)** it follows that at high temperatures (energies) $(\boldsymbol{\Delta U})_{meas}$ could hardly be a **primarily measurable** thermodynamic quantity. Because of this, it is expedient to use a counterpart of **Definition 2**.

Definition 4. Generalized Measurability in Thermodynamics
Any physical quantity in thermodynamics may be referred to as **generalized-measurable** or, for simplicity, **measurable** if any of its values may be obtained in terms of the **Primary Thermodynamic Measurability** of **Definition 3**.

In this way $(\Delta U)_{meas}$ from the formula **(28g)** is a **measurable** quantity.

Based on the preceding formulae, it is clear that we have the limiting transition

$$\boldsymbol{(\Delta U)_{meas}(T \approx T_{max}) \stackrel{(N_{\Delta(1/T)} \approx 1) \to (N_{\Delta(1/T)} \gg 1)}{\to} (\Delta U)_{meas}(T \ll T_{max} \propto T_p),}$$

that is analogous to the corresponding formula in a quantum theory.

Therefore, in this case the analog of *Comment 2*.* in Subsection 2.2 is valid.

Comment 2* Thermodynamics

From the above formulae it follows that, within ***GUPT*** *, the* **primarily measurable** *variations (quantities) are derived, to a high accuracy, from the* **generalized-measurable** *variations (quantities) only in the low-temperature limit* $T \ll T_{max} \propto T_p$.

R2.1 It is obvious that all the calculations associated with **measurability** of inverse temperature $\frac{1}{T}$ are valid for $\beta = \frac{1}{k_B T}$ as well. Specifically, introducing $\beta_{min} = \tilde{\beta} = \tilde{\tau}/k_B$, we can rewrite all the corresponding formulae in the ''**measurable**'' variant replacing $1/T$ ($\Delta(1/T)$) by β,$\tilde{\tau}$ by $\tilde{\beta}$ and retaining $N_{1/T}$ ($N_{\Delta(1/T)}$).

R2.2. Naturally, the problem of compatibility between the **measurability** definitions in quantum theory and in thermodynamics arises: is there any contradiction between **Definition 1** from Subsection 2.2 and **Definitions 3** from Subsection 2.3 ?

On the basis of the previous formulae we can state:

***measurability** in quantum theory and **thermodynamic measurability** are completely compatible and consistent as the minimal unit of inverse temperature* $\tilde{\tau}$ *is nothing else but the minimal time* $t_{min} = \tau$ *up to a constant factor. And hence* $N_{1/T}, (N_{\Delta(1/T)})$ *is nothing else but* $N_t, (N_{\Delta t})$*. Then it is clear that* $N_t = N_{a=tc}$.

R2.3 Finally, from the above formulae it follows that the **measurable** temperature T is varying as follows:

$$\boldsymbol{T = \frac{T_{max}}{N_{1/T}}, T = T_{max} \propto T_p, N_{1/T} \gg 1;}$$

$$\boldsymbol{T = \frac{T_{max}}{1/2(N_{1/T} + \sqrt{N_{1/T}^2 - 1})}, T \approx T_{max} \propto T_p, N_{1/T} \approx 1.} \qquad \textbf{(28h)}$$

In such a way **measurable** temperature is a **discrete quantity** but at low energies it is almost constantly varying – so, the theoretical calculations are very similar to those of the well-known continuous theory. In the reality, discreteness manifests itself in the case of high energies only.

Black Holes and Measurability

Now let us show the applicability this results to a quantum theory of black holes. Consider the case of Schwarzschild's black hole. It seems logical to support the idea suggested in the Introduction to the recent overview presented by seven authors

Gerard 't Hooft, Steven B. Giddings, Carlo Rovelli, Piero Nicolini, Jonas Mureika, Matthias Kaminski, and Marcus Bleicher, The Good, the Bad, and the Ugly of Gravity and Information, [arXiv:1609.01725v1 [hep-th] 6 Sep 2016]:

''Since for (asymptotically flat Schwarzschild) black holes the temperatures increase as their masses decrease, soon after Hawking's discovery, it became clear that a complete description of the evaporation process would ultimately require a consistent quantum theory of gravity. This is necessary as the semiclassical formulation of the emission process breaks down during the final stages of the evaporation as characterized by Planckian values of the temperature and spacetime curvature''.

Naturally, it is important to study the transition from low to high energies in the indicated case.
In this Section consideration is given to gravitational dynamics at low $E \ll E_p$ and at high $E \approx E_p$ energies in the case of the Schwarzschild black hole and in a more general case of the space with static spherically-symmetric horizon in space-time in terms of **measurable quantities** from the previous Section.
It should be noted that such spaces and even considerably more general cases have been thoroughly studied from the viewpoint of gravitational thermodynamics in remarkable works of professor T. Padmanbhan

The case of a static spherically-symmetric horizon in space-time is considered, the horizon being described by the metric

$$ds^2 = -f(r)c^2dt^2 + f^{-1}(r)dr^2 + r^2d\Omega^2. \qquad \textbf{(29)}$$

The horizon location will be given by a simple zero of the function $f(r)$, at the radius $r = a$.
Then at the horizon $r = a$ Einstein's field equations

$$\frac{c^4}{G}\left[\frac{1}{2}f'(a)a - \frac{1}{2}\right] = 4\pi P a^2 \quad \textbf{(30a)}$$

where $P = T_r^r$ is the trace of the momentum-energy tensor and radial pressure.
Therewith, the condition $f(a) = 0$ and $f'(a) \neq 0$ must be fulfilled.

On the other hand it is known that for horizon spaces one can introduce the temperature that can be identified with an analytic continuation to imaginary time. In the case under consideration

$$\boldsymbol{k_B T = \frac{\hbar c f'(a)}{4\pi}}. \qquad \textbf{(31)}$$

It is shown that in the initial (continuous) theory the Einstein Equation for horizon spaces in the differential form may be written as a thermodynamic identity (the first principle of thermodynamics)

$$\underbrace{\frac{\hbar c f'(a)}{4\pi}}_{k_B T} \underbrace{\frac{c^3}{G\hbar} d\left(\frac{1}{4} 4\pi a^2\right)}_{dS} \underbrace{- \frac{1}{2}\frac{c^4 da}{G}}_{-dE} = \underbrace{P d\left(\frac{4\pi}{3} a^3\right)}_{P\,dV}. \qquad \textbf{(30b)}$$

where, as noted above, T -- temperature of the horizon surface, S -- corresponding entropy, E-- internal energy, V -- space volume.
It is impossible to use **(30b)** in the formalism under consideration because, as follows from the given results da, dS, dE, dV are not **measurable quantities**.

First, we assume that a value of the radius r at the point a is a **primarily measurable quantity** in the sense of **Definition 1** i.e. $a = a_{meas} = N_a \ell$, where $N_a > 0$ - integer, and the temperature T from the left-hand side of **(31)** is the **measurable** temperature $T = T_{meas}$ in the sense of **Definition 3.**

Then, in terms of **measurable** quantities, first we can rewrite **(30a)** as

$$\frac{c^4}{G}\left[\frac{2\pi k_B T}{\hbar c} a_{meas} - \frac{1}{2}\right] = 4\pi P a_{meas}^2. \qquad \textbf{(30c)}$$

We express $a = a_{meas}$ in terms of the deformation parameter α_a (formula as

$$a = \ell \alpha_a^{-1/2}; \qquad \textbf{(30 } \alpha\textbf{)}$$

the temperature T is expressed in terms of $T_{max} \propto T_p$.

Then, considering that $T_p = E_p / k_B$, equation **(30c)** may be given as

$$\frac{c^4}{G}\left[\frac{\pi E_p}{\sqrt{\alpha'} N_{1/T} \hbar c} \ell \alpha_a^{1/2} - \frac{1}{2}\alpha_a\right] = 4\pi P \ell^2. \quad \textbf{(30d)}$$

Because $\ell = 2\sqrt{\alpha'} l_p$ and $l_p = \frac{\hbar c}{E_p}$, we have

$$\frac{c^4}{G}\left[\frac{2\pi E_p}{N_{1/T}\hbar c} l_p \alpha_a^{1/2} - \frac{1}{2}\alpha_a\right] = \frac{c^4}{G}\left[\frac{2\pi}{N_{1/T}} \alpha_a^{1/2} - \frac{1}{2}\alpha_a\right] = 4\pi P \ell^2.$$

(30e)

Note that in its initial form this equation has been considered in a continuous theory, i.e. at low energies $E \ll E_p$. Consequently, in the present formalism it is implicitly meant that the ''measurable counterpart'' of this equation also initially considered at low energies, in particular, $N_{1/T} \gg 1$.

Let us consider the possibility of generalizing (to high energies) taking two different cases.

Measurable case for low energies: $E \ll E_p$. Then $a = a_{meas} = N_a \ell$, where the integer number is $N_a \gg 1$ or similarly $N_{1/T} \gg 1$. In this case GUP, to a high accuracy, is extended to HUP.
As this takes place, $\alpha_a = \alpha_a(HUP)$ is a **primarily measurable** quantity (**Definition 1**), $\alpha_a \approx N_a^{-2}$, though taking a discrete series of values but varying smoothly, in fact *continuously*. **(30e)** is a quadratic equation with respect to $\alpha_a^{1/2} \approx N_a^{-1}$ with the two parameters $N_{1/T}^{-1}$ and P. In this terms, the equation **(30e)** may be rewritten as

$$\boldsymbol{\frac{c^4}{G}\left[\frac{2\pi}{N_{1/T}}\alpha_a^{1/2}(HUP) - \frac{1}{2}\alpha_a(HUP)\right] = 4\pi P\ell^2.} \quad \textbf{(30f)}$$

So, at low energies the equation **(30e)** (or **(30f)**) written for the discretely-varying $\boldsymbol{\alpha_a}$ may be considered in a continuous theory.

As a result, in the case under study we can use the basic formulae from a continuous theory considering them valid to a high accuracy.

In particular, in the notation used for ***Schwarzschild's black hole***, we have

$$\boldsymbol{r_s = N_a \ell = \frac{2GM}{c^2}; M = \frac{N_a \ell c^2}{2G}}. \qquad \textbf{(31.BH1)}$$

As its temperature is given by the formula

$$\boldsymbol{T_H = \frac{\hbar c^3}{8\pi G M k_B}}, \qquad \textbf{(31.BH2)}$$

at once we get

$$\boldsymbol{T_H = \frac{\hbar c}{4\pi k_B N_a \ell} = \frac{\hbar c \alpha_a^{1/2}}{4\pi k_B \ell}}. \qquad \textbf{(31.BH3)}$$

Comparing this expression to the expression with high $N_{1/T}$ ($N_{1/T} \gg 1$) for temperature, we can find that at low energies $E \ll E_p$, due to comment **R2.2.** from Subsection 2.3, the number $N_{1/T}$ is actually coincident with the number N_a:

$$\boldsymbol{N_{1/T} = N_a = \alpha_a^{-1/2} (HUP)}. \ \textbf{(31.BH4)}$$

The substitution of the last expression into the quadratic equation **(30f)** for $\alpha_a^{1/2}$ makes it a linear equation for α_a with a single parameter P.

Measurable case for high energies: $E \approx E_p$. Then, a is the **generalized measurable** quantity $a = a_{meas} = 1/2(N_a + \sqrt{N_a^2 - 1})\ell$, with the integer $N_a \approx 1$.

The quantity

$$\boldsymbol{\Delta a_{meas}(q) = \frac{1}{2\left(N_a+\sqrt{N_a^2-1}\right)\ell} - N_a\ell =}$$

$$= \boldsymbol{1/2(\sqrt{N_a^2 - 1} - N_a)\ell} \qquad \textbf{(32)}$$

may be considered as a **quantum correction** for the **measurable** radius $r = a_{meas}$, that is infinitesimal at low energies $E \ll E_p$ and not infinitesimal for high energies $E \approx E_p$.

In this case there is no possibility to replace **GUP** by **HUP**. In equation **(30e)** $\alpha_a = \alpha_a(GUP)$ is a **generalized measurable** quantity (**Definition 2**).
As noted in Comment **R2.3**, in this case the number $N_{1/T}$ is replaced by $1/2(N_{1/T} + \sqrt{N_{1/T}^2 - 1})$, i.e. the equation is of the form

$$\frac{c^4}{G}\left[\frac{2\pi}{1/2(N_{1/T}+\sqrt{N_{1/T}^2-1})}\alpha_a^{1/2}(GUP) - \frac{1}{2}\alpha_a(GUP)\right] = 4\pi P\ell^2.$$

(33)

In so doing the theory becomes really discrete, and the solutions of **(33)** take a discrete series of values for every N_a or ($\alpha_a(GUP)$) sufficiently close to 1.

In this formalism for a ''quantum'' Schwarzschild's black hole (i.e. at high energies $E \approx E_p$) formula **(31.BH3)** is replaced by

$$T_H(Q) = \frac{\hbar c}{\pi k_B(N_a+\sqrt{N_a^2-1})\ell} = \frac{\hbar c\alpha_a^{1/2}(GUP)}{4\pi k_B\ell}. \qquad \textbf{(31.BH3Q)}$$

We should make several remarks which are important.

Remark 3.3.
The parameter $\alpha_a = \alpha_a(HUP)$, within constant factors, is coincident with the Gaussian curvature $\boldsymbol{K_a}$ corresponding to **primary measurable** $a = N_a \ell$:

$$\boldsymbol{\alpha_a = \frac{\ell^2}{a^2} = \ell^2 K_a}. \quad \textbf{(Gauss1)}$$

Because of this, the transition from $\boldsymbol{\alpha_a(HUP)}$ to $\boldsymbol{\alpha_a(GUP)}$ may be considered as a basis for **"quantum corrections"** to the Gaussian curvature K_a in the high-energy region $E \approx E_p$:

$$\boldsymbol{\alpha_a(GUP) - \alpha_a(HUP) = \ell^2 [\frac{1}{1/4(N_a + \sqrt{N_a^2 - 1})^2 \ell^2} - \frac{1}{N_a^2 \ell^2}] =} \quad \textbf{(Gauss1.c)}$$
$$\boldsymbol{= \ell^2 (K_a^Q - K_a)},$$

where the "measurable quantum Gaussian curvature " K_a^Q is defined as

$$\boldsymbol{K_a^Q = \frac{1}{1/4(N_a + \sqrt{N_a^2 - 1})^2 \ell^2}}. \qquad \textbf{(Gauss1.Q)}$$

In a similar way, we can derive a **"measurable quantum correction "** for the mass $\boldsymbol{M}$ of a Schwarzschild black hole at high energies.

Remark 3.4.
A minimal value of $\boldsymbol{N_a = 1}$ is ***unattainable*** because in this case obtain a value of the length l that is below the minimum $l < l$ for the momenta and energies above the maximal ones, and that is impossible.
Thus, we always have $\boldsymbol{N_a \geq 2}$.

Remark 3.5. It is clear that we have the following transition:

$$\boldsymbol{Eq.(33)(E \approx E_p) \overset{(N_a \approx 1) \rightarrow (N_a \gg 1)}{\rightarrow} Eq.(30f)(E \ll E_p)}$$

.

Remark 3.6. So, all the members of the gravitational equation apart from $\boldsymbol{P}$, are expressed in terms of the measurable parameter $\boldsymbol{\alpha_a}$**.** From this it follows that P should be also expressed in terms of the measurable parameter α_a, i.e. $P = P(\alpha_a)$: $E \ll E_p$, $P = P[\alpha_a(HUP)]$ at low energies and $E \approx E_p$,$P = P[\alpha_a(GUP)]$ at high energies. Then, due to the above formulae, we can have for a ''quantum'' Schwarzschild black hole the “ horizon” gravitational equation in terms of **measurable** quantities

$$\boldsymbol{(4\pi - 1)\frac{c^4}{G}\alpha_a(GUP) = 8\pi P[\alpha_a(GUP)]\ell^2},$$

where $\alpha_a(GUP)$ takes a discrete series of the values $\alpha_a(GUP) = (1/2(N_a + \sqrt{N_a^2 - 1}))^{-2}$; $N_a \geq 2$ is a small integer.

Planck's Deformation of Basic Quantities

The results from the previous Section may be interpreted as follows: at high energies i.e. at Planck's scales $E \approx E_P$, all the basic quantities l, t, T, and so on written by **measurable** terms are modified (deformed).
In particular, we get

$$\begin{cases} \boldsymbol{l} \overset{(\boldsymbol{E}=\boldsymbol{E_p})\rightarrow(\boldsymbol{E}\approx\boldsymbol{E_p})}{\rightarrow} \frac{1}{2}(\boldsymbol{l} + \sqrt{\boldsymbol{l}^2 - \boldsymbol{\ell}^2}) \\ \boldsymbol{t} \overset{(\boldsymbol{E}=\boldsymbol{E_p})\rightarrow(\boldsymbol{E}\approx\boldsymbol{E_p})}{\rightarrow} \frac{1}{2}(\boldsymbol{t} + \sqrt{\boldsymbol{t}^2 - \boldsymbol{\tau}^2}) \\ \frac{1}{\boldsymbol{T}} \overset{(\boldsymbol{E}=\boldsymbol{E_p})\rightarrow(\boldsymbol{E}\approx\boldsymbol{E_p})}{\rightarrow} \frac{1}{2}(\frac{1}{\boldsymbol{T}} + \sqrt{(\frac{1}{\boldsymbol{T}})^2 - \tilde{\boldsymbol{\tau}}^2}) \end{cases} \qquad \textbf{(P1)}$$

In a similar way, we can obtain the *high-energy* ($E \approx E_p$) **"measurable"** deformation for all the other physical quantities $P, E, U, \ldots$ initially specified at low energies $E = E_p$ in terms of **measurable quantities**.

Consequently, we can derive the **"measurable" quantum corrections"** Δ_Q for $l, t, 1/T$ and so on:

$$\begin{cases} \boldsymbol{\Delta_Q l = \frac{1}{2}(l + \sqrt{l^2 - \ell^2}) - l = \frac{1}{2}(\sqrt{l^2 - \ell^2} - l)} \\ \boldsymbol{\Delta_Q t = \frac{1}{2}(t + \sqrt{t^2 - \tau^2}) - t = \frac{1}{2}(\sqrt{t^2 - \tau^2} - \tau)} \\ \boldsymbol{\Delta_Q \frac{1}{T} = \frac{1}{2}(\sqrt{(\frac{1}{T})^2 - \tilde{\tau}^2} - \frac{1}{T})} \\ \ldots\ldots\ldots\ldots\ldots \end{cases} \quad \textbf{(P2)}$$

For for a **"quantum" Schwarzschild's black entropy**

$$\boldsymbol{S_{Schw,meas-q} = \frac{4\pi(\frac{1}{2}(N_{r_s} + \sqrt{N_{r_s}^2 - 1})\ell)^2}{\ell^2/\alpha'} =} \quad \textbf{(P3)}$$

$$\boldsymbol{= \pi\alpha'(N_{r_s} + \sqrt{N_{r_s}^2 - 1})^2.}$$

And a "**measurable quantum correction**" for the temperature $\boldsymbol{T}$, for the mass $\boldsymbol{M}$ and entropy $\boldsymbol{S}$ of a Schwarzschild black hole:

$$\begin{cases} \Delta_Q T = T_H(Q) - T_H = \frac{\hbar c}{4\pi k_B \ell}(\alpha_{r_s}^{1/2}(GUP) - \alpha_{r_s}^{1/2}(HUP)), \\ \Delta_Q M = M(Q) - M = \frac{\ell c^2}{2G}(\alpha_{r_s}^{-1/2}(GUP) - \alpha_{r_s}^{-1/2}(HUP)) = \\ = \frac{\ell c^2}{4G}\left(\sqrt{N_{r_s}^2 - 1} - N_{r_s}\right) = \frac{c^2}{4G}\left(\sqrt{r_s^2 - \ell^2} - r_s\right), \text{(P2S)} \\ \Delta_Q S = S_{Schw,meas-q} - S_{Schw,meas} = \\ = \pi\alpha'(N_{r_s} + \sqrt{N_{r_s}^2 - 1})^2 - 4\pi\alpha' N_{r_s}^2 = \\ = \pi\alpha'\left(2N_{r_s}\sqrt{N_{r_s}^2 - 1} - 2N_{r_s}^2 - 1\right). \end{cases}$$

As indicated by the last formulae the **measurable quantum corrections** *are nothing else but the difference between the* **generalized measurable** *quantities and the* **primarily measurable** *quantities.*

Conclusion

In the general case the problem at hand is as follows:
the formulation of Gravity in terms of **measurable** *quantities and also the derivation of a solution in terms of* **measurable** *quantities*.

This **''measurable''** Gravity -- discrete theory that is practically continuous at low energies $E \ll E_p$ and very close to the Einstein theory, though with some principal differences. By author's opinion, in the low-energy **''measurable''** variant of Gravity we should have no solutions without physical meaning, specifically ***Godel's solution***.

At high energies $E \approx E_p$ this ''measurable'' Gravity should be really a discrete theory enabling the transition to the low-energy ''measurable'' variant of Gravity. Still it is obvious that, to construct a measurable variant of Gravity at all the energy scales, in the general case we need both the primarily measurable variations $\Delta p(N_{\Delta x}, HUP)$ and the generalized-measurable variations $\Delta p(N_{\Delta x}, GUP)$ from formula. The author believes that such construction should be realized jointly with a construction of a measurable variant for Quantum Theory (QT).

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